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A canonical Frobenius structure

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Abstract We show that it makes sense to speak of the Frobenius manifold attached to a convenient and nondegenerate Laurent polynomial.

1 Introduction

This paper is the last of a series devoted to the construction of Frobenius structures on the base of a deformation of a convenient and nondegenerate Laurent polynomial f, defined on the torus $U = (\mathbb{C}^*)^n$. The motivations and the general setting are given in [4] where such a construction (an imitation of the one given by Saito [14] in the case of germs) is done starting with a universal unfolding of f. Two major difficulties arise: first, the analysis of the Fourier–Laplace transform of the Brieskorn lattice of this unfolding leads to a transcendent process of analytization in the variables of the polynomial, because of the critical points vanishing at infinity of the deformed polynomials in the unfolding (see [4, Sect. 2]). Second, the universality condition means that the Kodaira Spencer map is an isomorphism and it is not known, unlike germs, if any unfolding is induced by a universal one. Therefore, two different (universal) unfoldings could produce two different Frobenius structures which are not easy to compare.

In some cases, these difficulties are overcome: in [5], examples are given using the fact that semisimple and simply connected Frobenius manifolds are determined by a finite set of numbers, and this is a result of Dubrovin [6]. More generally, we have explained in [2] how one can construct, using a result of Hertling and Manin [8], Frobenius structures which are determined by a restricted set of algebraic data (the "initial conditions"). We point out the fact that this construction simplifies greatly the one in [4] because we only consider deformations of f that do not produce critical points vanishing at infinity. However, these initial conditions are not unique and, starting from f, it is a priori possible to construct several

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Frobenius structures. The goal of this paper is to compare them, in fact to show that they are all isomorphic. Finally, we attach a *canonical* Frobenius structure, which is determined by a restricted set of algebraic data, to (almost) any convenient and nondegenerate Laurent polynomial. The point is that our initial conditions are in essence as simple to compute as the ones in [6]. This is especially useful if one wants to compare the Frobenius structures constructed in this paper with the ones coming from different fields of mathematics and which are also determined by a restricted set of data: it is enough to identify these data and to compare them. This approach has been used in [1].

Let us precise the situation: let $F : U \times \mathbb{C}^r \to \mathbb{C}$ be the subdiagram deformation of f defined by

$$F(u, x) = f(u) + \sum_{i=1}^{r} x_i g_i(u),$$

where the g_i's are some Laurent polynomials (we put $x = (x_1, \ldots, x_r)$ and $u = (u_1, \ldots, u_n)$). Here, subdiagram means that the Laurent polynomials $g_1, \ldots, g_r$ are linear combinations of monomials $u_1^{a_1} \ldots u_n^{a_n}$ where $a = (a_1, \ldots, a_n)$ belongs to the interior of the Newton polyhedron of f (we will also say that $g_1, \ldots, g_r$ are subdiagram Laurent polynomials). We attach to F a Frobenius type structure on $\mathbb{A}^r$, that is a t-uple

$$\mathbb{F} = (\mathbb{A}^r, E, \bigtriangledown, R_0, R_\infty, \Phi, g),$$

where E is a free $\mathbb{C}[x]$-module, Φ a Higgs field, $\bigtriangledown$ a flat connection on E, g a metric, R_0 and R_∞ two endomorphisms of E, these different objects satisfying some natural compatibility relations. This is the initial condition and it is obtained by solving the Birkhoff problem for the Brieskorn lattice of F (see Sect. 4). Once $\mathbb{F}$ is fixed, and up to the existence of a pre-primitive and homogeneous form, that is a $\bigtriangledown$-flat section ω of E satisfying an injectivity condition (IC), a generation condition (GC) and a homogeneity condition (H), we can, following Hertling and Manin [8], unfold $\mathbb{F}$ and equip $(\mathbb{C}^\mu, 0)$ with a Frobenius structure where μ is the global Milnor number of f. Notice that the unfolding process given in *loc. cit.* is analytic but now this affects only the parameters of the deformation.

In this paper, we will take for ω the class of the volume form $\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}$ in E, the reason being that ω is the $\bigtriangledown$-flat extension to E of the canonical primitive form attached to f by [4, Sect. 4.d]. Then ω satisfies the condition (IC) at least if the g_i's are $\mathbb{C}$-linearly independent, in which case we will say that the subdiagram deformation F is *injective*, and condition (H) follows from the homogeneity of the canonical primitive form attached to f by *loc. cit.* Let us have a closer look at (GC): ω will satisfy (GC) if any element of A_f, the Jacobi algebra of f, can be written as the class of a polynomial in $g_1, \ldots, g_r, f$ with coefficients in $\mathbb{C}$. Of course, this will be true if any element of A_f can be written as the class of a polynomial in $g_1, \ldots, g_r$ with coefficients in $\mathbb{C}$, in which case we will say that $(g_1, \ldots, g_r)$ is a *lattice* in A_f, or if any element of A_f can be written as the class of a polynomial in f with coefficients in $\mathbb{C}$. The latter case occurs when the multiplication by f on A_f is *regular*, in particular if the critical values of f are all distinct: this is the framework of Dubrovin [6]. We focus now on the former case: let $(g_1, \ldots, g_r)$ be a lattice in A_f. Then ω is pre-primitive and homogeneous but the desired Frobenius structure will depend a priori on the lattice $(g_1, \ldots, g_r)$: two different lattices (made with subdiagram Laurent polynomials) could give two distinct Frobenius manifolds. We show:

Theorem 1 *Let f be a convenient and nondegenerate Laurent polynomial, μ its global Milnor number. Assume that there exist subdiagram Laurent polynomials $g_1, \ldots, g_r$ such* *that* $(g_1, \ldots, g_r)$ *is a lattice in* A_f. *Then the construction of Hertling and Manin equips* $(\mathbb{C}^\mu, 0)$ *with a canonical Frobenius structure. Up to isomorphism, this Frobenius structure does not depend on the lattice* $(g_1, \ldots, g_r)$.

Theorem 1 includes also the regular case: if moreover the multiplication by f is regular, it follows from the discussion above that there are at least two ways to construct Frobenius structures. They will be isomorphic. Theorem 1 can be used to define canonical CDV and tt^* structures (see [7,13]) attached to a convenient and nondegenerate Laurent polynomial.

Up to a slightly stronger generation condition, we can give a global counterpart of Theorem 1. Let F as above be an injective subdiagram deformation of f, A_F its Jacobi algebra, which is a $\mathbb{C}[x]$-module of finite type. We will say that ω satisfies $(GC)^{gl}$ for F if $(g_1, \ldots, g_r)$ is a lattice in A_F, that is if any element of A_F can be written as (the class of) a polynomial in $g_1, \ldots, g_r$ with coefficients in $\mathbb{C}[x]$.

Theorem 2 *Let* $a \in \mathbb{C}^r$ *and assume that* ω *satisfies* $(GC)^{gl}$ *for* F. *Then,*

(1) *the canonical Frobenius structure attached by Theorem 1 to the convenient and nondegenerate Laurent polynomial* $F_a := F(., a)$ *is obtained by an analytic continuation of the one attached to* f,
(2) *for any injective and subdiagram deformation G of* f, *the canonical Frobenius structure attached by Theorem 1 to the convenient and nondegenerate Laurent polynomial* $G_a := G(., a)$ *is obtained by an analytic continuation of the one attached to* f.

Theorems 1 and 2 are detailed in Sect. 7.

This paper is organized as follows: in Sect. 2, we recall the basic facts about the Frobenius type structures and their deformations. In Sect. 3, we explain the construction of Hertling and Manin. Then we apply all this to a geometric situation: we define the canonical Frobenius type structures attached to a subdiagram deformation of a convenient and nondegenerate Laurent polynomial (Sect. 4) and the canonical pre-primitive form (Sect. 5). In Sect. 6 we study the existence of universal deformations of the canonical Frobenius type structure. We show in particular that one can define global universal deformations along the space of the subdiagram monomials. Sect. 7 is devoted to the proof of Theorems 1 and 2. Last, we summarize the results and give our recipe to construct Frobenius manifolds in Sect. 8. We end with an example.

Notations. In this paper we will put $U = (\mathbb{C}^*)^n$, $u = (u_1, \ldots, u_n)$, $x = (x_1, \ldots, x_r)$,

$$K = \mathbb{C}[u, u^{-1}] = \mathbb{C}[u_1, \ldots, u_n, u_1^{-1}, \ldots, u_n^{-1}]$$

and

$$\frac{du}{u} = \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}.$$

If f is a Laurent polynomial, A_f will denote its Jacobi algebra

$$\frac{K}{\left(\frac{\partial f}{\partial u_1}, \ldots, \frac{\partial f}{\partial u_n}\right)}.$$

2 Frobenius type structure

2.1 Frobenius type structure on a complex analytic manifold

Let M be a complex analytic manifold. Let us be given a t-uple

$$(M, E, \bigtriangledown, R_0, R_\infty, \Phi, g),$$

where

- E is a locally free $\mathcal{O}_M$-module,
- R_0 and R_∞ are $\mathcal{O}_M$-linear endomorphisms of E,
- $\Phi : E \to \Omega^1_M \otimes E$ is an $\mathcal{O}_M$-linear map,
- g is a *metric* on E, i.e. a $\mathcal{O}_M$-bilinear form, symmetric and nondegenerate,
- $\bigtriangledown$ is a connection on E.

Definition 2.1.1 The t-uple

$$(M, E, \bigtriangledown, R_0, R_\infty, \Phi, g)$$

is a Frobenius type structure on M if the following relations are satisfied:

$$\begin{aligned}
&\bigtriangledown^2 = 0, \bigtriangledown(R_\infty) = 0, \Phi \wedge \Phi = 0, [R_0, \Phi] = 0,\\
&\bigtriangledown(\Phi) = 0, \bigtriangledown(R_0) + \Phi = [\Phi, R_\infty],\\
&\bigtriangledown(g) = 0, \Phi^* = \Phi, R_0^* = R_0, R_\infty + R_\infty^* = rId
\end{aligned}$$

for a suitable constant r, where * denotes the adjoint with respect to g.

We will use systematically the following lemma, which is a direct consequence of the definition:

Lemma 2.1.2 *Let $(M, E, \bigtriangledown, R_0, R_\infty, \Phi, g)$ be a Frobenius type structure on M. Then*:

(1) *the connection $\bigtriangledown$ is flat.*
(2) *Let ε be a $\bigtriangledown$-flat basis of E, $C = \sum_i C^{(i)} dx_i$ (resp. B_0, B_∞) the matrix of Φ (resp. R_0, R_∞) in this basis. One has, for all i and for all j,*

$$\begin{aligned}
&\frac{\partial C^{(i)}}{\partial x_j} = \frac{\partial C^{(j)}}{\partial x_i},\\
&[C^{(i)}, C^{(j)}] = 0,\\
&[B_0, C^{(i)}] = 0,\\
&C^{(i)} + \frac{\partial B_0}{\partial x_i} = [B_\infty, C^{(i)}],\\
&C^{(i)*} = C^{(i)}, B_0^* = B_0, B_\infty + B_\infty^* = rI
\end{aligned}$$

(I is the identity matrix). The matrix B_∞ is constant.

Remark 2.1.3 (1) A Frobenius type structure on a point is a t-uple (E, R_0, R_∞, g) where E is a finite dimensional $\mathbb{C}$-vector space, R_0 and R_∞ are endomorphisms of E, and g is a bilinear, symmetric and nondegenerate form on E such that

$$R_0^* = R_0, R_\infty + R_\infty^* = rId$$

for a suitable constant $r \in \mathbb{C}$. As above, * denotes the adjoint with respect to g.

(2) We will also consider Frobenius type structures on $\mathbb{A}^r$ that is t-uples

$$(\mathbb{A}^r, E, \bigtriangledown, R_0, R_\infty, \Phi, g),$$

where E is a free $\mathbb{C}[x]$-module. The objects $\bigtriangledown$, R_0, R_∞, Φ and g are defined as above (replace $\mathcal{O}_M$-linear by $\mathbb{C}[x]$-linear) and satisfy the relations of Definition 2.1.1.

2.2 Construction of Frobenius type structures

2.2.1 From Frobenius type structures to flat connections

Let M be a complex analytic manifold and E be locally free $\mathcal{O}_M$-module. Let $\pi : \mathbb{P}^1 \times M \to M$ be the projection, $\mathcal{E} := \pi^* E$ and ∇ the meromorphic connection on $\mathcal{E}$ defined by

$$\nabla = \pi^* \bigtriangledown + \tau \pi^* \Phi - (\tau R_0 + R_\infty) \frac{d\tau}{\tau},$$

where τ is the coordinate on the chart centered at infinity. Then ∇ is flat if and only if the t-uple

$$(M, E, \bigtriangledown, R_0, R_\infty, \Phi)$$

is a Frobenius type structure on M (without metric).

2.2.2 From flat connections to Frobenius type structures

Conversely, a trivial bundle $\mathcal{E}$ on $\mathbb{P}^1 \times M$ equipped with a flat connection ∇, with logarithmic poles along $\{\infty\} \times M$ and with poles of order 1 along $\{0\} \times M$, enables us to construct a Frobenius type structure (without metric)

$$(M, E, \bigtriangledown, R_0, R_\infty, \Phi),$$

where $E := \mathcal{E}_{|\{0\}\times M}$ (see for instance [12, Chap. VII] for the details). One can also get in this way a Frobenius type structure

$$(M, E, \bigtriangledown, R_0, R_\infty, \Phi, g)$$

with metric (see [12, Chap. VI, 2.b]). All Frobenius type structures that we will consider are constructed in this way (see Sect. 4.3).

2.3 Deformations of Frobenius type structures

Since one knows how to define the pullback of a bundle equipped with a connection, one can define, using Sect. 2.2, the pullback of a Frobenius type structure: if $\psi : N \to M$ where M and N are two complex analytic manifolds and if $\mathcal{F}$ is a Frobenius type structure on M then $\psi^*\mathcal{F}$ is a Frobenius type structure on N.

Definition 2.3.1 (1) If ψ is a closed immersion, one says that $\mathcal{F}$ is a *deformation* of $\psi^*\mathcal{F}$.
(2) Two deformations of a same Frobenius type structure are *isomorphic* if one comes from the other by a base change inducing an isomorphism on the corresponding tangent bundles.
(3) Let $\mathcal{F}$ be a Frobenius type structure on N. A deformation $\tilde{\mathcal{F}}$ of $\mathcal{F}$ is *universal* if any other deformation of $\mathcal{F}$ comes from $\tilde{\mathcal{F}}$ by a unique base change, inducing the identity on N.

If it exists, a universal deformation is unique, up to isomorphism.

3 Hertling and Manin's theorem. Construction of Frobenius manifolds

3.1 Pre-primitive forms

3.1.1 The analytic case

Let $\mathcal{F} = (M, E, \bigtriangledown, R_0, R_\infty, \Phi, g)$ be a Frobenius type structure on M. Suppose first that M is a punctual germ of a complex analytic manifold. Let ω be a $\bigtriangledown$-flat section of E.

Definition 3.1.1 The period map attached to ω is the map

$$\varphi_\omega : \Theta_M \to E, \tag{1}$$

$$\xi \mapsto -\Phi_\xi(\omega). \tag{2}$$

This period map can be seen as a $\bigtriangledown$-flat differential form: in coordinates,

$$\varphi_\omega = -\sum_{i=1}^{r} \Phi_{\partial_{x_i}}(\omega) dx_i.$$

Assume moreover that $\omega = \varepsilon_1$ where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_\mu)$ is a $\bigtriangledown$-flat basis of E. With the notations of Lemma 2.1.2, one then gets

$$\varphi_\omega = -\sum_{j=1}^{\mu} \left(\sum_{i=1}^{r} C_{j1}^{(i)}(x) dx_i \right) \varepsilon_j.$$

Lemma 2.1.2 shows also that the differential form $\sum_{i=1}^{r} C_{j1}^{(i)}(x) dx_i$ is d-closed : let Γ_{j1} be the function such that $\Gamma_{j1}(0) = 0$ and $d\Gamma_{j1}(x) = \sum_{i=1}^{r} C_{j1}^{(i)}(x) dx_i$. Define

$$\chi_\omega^\varepsilon : M \to E, \tag{3}$$

$$x \mapsto \sum_{j=1}^{\mu} \Gamma_{j1}(x) \varepsilon_j. \tag{4}$$

The basis ε being fixed, χ_ω^ε can also be seen as a map

$$\chi_\omega^\varepsilon : M \to \mathbb{C}^\mu \tag{5}$$

$$x \mapsto (\Gamma_{11}(x), \dots, \Gamma_{\mu 1}(x)) \tag{6}$$

Definition 3.1.2 χ_ω^ε is the primitive map attached to the $\bigtriangledown$-flat section ω and to the basis ε.

Remark 3.1.3 Up to isomorphism, the map χ_ω^ε does not depend on the basis ε. We will omit the index ε : there will be no confusion because we will always work with M. Saito's canonical basis (see Sect. 4.3).

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_M$. The index ^{o} will denote the operation "modulo $\mathfrak{m}$".

Definition 3.1.4 Let ω be a $\bigtriangledown$-flat section of E. One says that ω is *pre-primitive* if

(GC) ω^o and its images under the iteration of the maps R_0^o and Φ_ξ^o (for all ξ) generate E^o,
(IC) $\varphi_\omega^o : \Theta_M^o \to E^o$ is injective.

Remark 3.1.5 (1) If $M = \{point\}$ the condition (IC) is empty. Assume moreover that R_0 is regular (i.e. its characteristic polynomial is equal to its minimal polynomial): there exists ω such that

$$\omega, R_0(\omega), \dots, R_0^{\mu-1}(\omega)$$

is a basis of E over $\mathbb{C}$ and ω is thus pre-primitive.

(2) If (GC) is satisfied, it is also satisfied in the neighborhood of 0: E is then generated by ω and its images under iteration of the maps R_0 and Φ_ξ (for all ξ).

Let now M be a simply connected complex analytic manifold. The period map attached to the ∇-flat section ω is the $\mathcal{O}_M$-linear map defined as in Definition 3.1.1. One defines also the primitive map χ_ω^ε, attached to the ∇-flat section ω and to the basis ε : since M is simply connected, χ_ω^ε is holomorphic on M. The definition of the pre-primitive forms depends now on the origin: if $a \in M$, $\mathfrak{m}^a$ will denote the maximal ideal of $\mathcal{O}_{M,a}$ and the index ^{a} the operation "modulo $\mathfrak{m}^a$".

Definition 3.1.6 Let ω be a ∇-flat section of E, $a \in M$. We will say that ω^a satisfies (GC) if ω^a and its images under the iteration of the maps R_0^a and Φ_ξ^a (for all ξ) generate E^a and that ω^a satisfies (IC) if

$$\varphi_\omega^a : \Theta_M^a \to E^a$$

is injective. One says that ω is *pre-primitive* for the origin a if ω^a satisfies (GC) and (IC).

3.1.2 The algebraic case

Let $\mathbb{F} = (\mathbb{A}^r, E, \nabla, R_0, R_\infty, \Phi, g)$ be a Frobenius type structure on $\mathbb{A}^r$. The period map attached to ω is now a $\mathbb{C}[x]$-linear map, defined on the Weyl algebra $\mathbb{A}^r(\mathbb{C}) = \mathbb{C}[x]\langle\partial_x\rangle$,

$$\varphi_\omega : \mathbb{A}^r(\mathbb{C}) \to E, \tag{7}$$

$$\xi \mapsto -\Phi_\xi(\omega). \tag{8}$$

The index ^{a} will denote the operation "modulo $(x-a)$".

Definition 3.1.7 Let ω be a ∇-flat section of E.

(1) We will say that ω satisfies the condition $(GC)^{gl}$ if ω and its images under the iteration of the maps R_0 and Φ_ξ (for all ξ) generate the $\mathbb{C}[x]$-module E and that ω satisfies the condition $(IC)^{gl}$ if

$$\varphi_\omega : \mathbb{A}^r(\mathbb{C}) \to E$$

is injective. We will say that ω is *globally pre-primitive* if ω satisfies $(GC)^{gl}$ and $(IC)^{gl}$.

(2) Let $a \in \mathbb{A}^r$. We will say that ω^a satisfies (GC) if ω^a and its images under the iteration of the maps R_0^a and Φ_ξ^a (for all ξ) generate E^a and that ω^a satisfies (IC) if φ_ω^a is injective. We will say that ω is pre-primitive for the origin a if ω^a satisfies (GC) and (IC).

Remark 3.1.8 (Analytization) A Frobenius type structure $\mathbb{F}$ on $\mathbb{A}^r$ gives, after analytization, a Frobenius type structure

$$\mathbb{F}^{\mathrm{an}} = (\mathbb{C}^r, E^{\mathrm{an}}, \nabla^{\mathrm{an}}, R_0^{\mathrm{an}}, R_\infty^{\mathrm{an}}, \Phi^{\mathrm{an}}, g^{\mathrm{an}})$$

on $\mathbb{C}^r$. Notice that E^{an} is canonically trivialized by a basis of (global) ∇-flat sections. A globally pre-primitive section ω of E gives a pre-primitive section ω^{an} of E^{an} for any choice of the origin.

3.2 Hertling and Manin's construction

Let $\mathcal{F} = (M, E, \bigtriangledown, R_0, R_\infty, \Phi, g)$ be a Frobenius type structure on a germ of complex analytic manifold M, ω a $\bigtriangledown$-flat section of E and χ_ω the primitive map attached to ω. If $\tilde{\mathcal{F}}$ is a deformation of $\mathcal{F}$, we will denote $\tilde{\chi}_\omega$ (resp. $\tilde{\varphi}_\omega$) the primitive map (resp. the period map) attached to the flat extension of ω. We will say that a $\bigtriangledown$-flat section of E is *homogeneous* if it is an eigenvector of R_∞. Frobenius structures are defined in [12, VII.2].

Theorem 3.2.1 (1) ([8, Theorem 2.5]) *Assume that the Frobenius type structure $\mathcal{F}$ has a pre-primitive section ω. Then $\mathcal{F}$ has a universal deformation. A deformation $\tilde{\mathcal{F}}$ of $\mathcal{F}$ is universal if and only if the primitive map (resp. period map) $\tilde{\chi}_\omega$ (resp. $\tilde{\varphi}_\omega$) is a diffeomorphism (resp. an isomorphism).*

(2) ([8, Theorem 4.5]) *A flat, pre-primitive and homogeneous section of the Frobenius type structure $\mathcal{F}$ defines, through the period map, a Frobenius structure on the base $\tilde{M}$ of any universal deformation of $\mathcal{F}$: $\tilde{M}$ is thus a Frobenius manifold.*

(3) *The Frobenius structures given by (2) on the bases of any two universal deformations are isomorphic.*

Proof (1) In brief, condition (GC) shows that one can construct deformations of the Frobenius type structure and condition (IC) is then used to show the universality of some of them (see also Remark 6.1.4 below).

(2) It follows from (1) that $\mathcal{F}$ has a universal deformation $\tilde{\mathcal{F}} = (\tilde{M}, \tilde{E}, \tilde{\bigtriangledown}, \tilde{R}_0, \tilde{R}_\infty, \tilde{\Phi}, \tilde{g})$. Moreover, the period map associated with the flat extension of the pre-primitive form is an isomorphism because the deformation is universal. One can thus carry the structures defined on $\tilde{E}$ onto $\Theta_{\tilde{M}}$, the sheaf of holomorphic vector fields on $\tilde{M}$, and gets, by definition, a (a priori nonhomogeneous) Frobenius structure on $\tilde{M}$. If moreover the pre-primitive form is homogeneous, its flat extension is also homogeneous because R_∞ carries flat sections onto flat sections: this gives the homogeneity of the Frobenius structure. This makes $\tilde{M}$ a Frobenius manifold.

(3) Let $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ be two universal deformations of $\mathcal{F}$, with bases $\tilde{M}$ and $\tilde{M}'$, $\tilde{\chi}_\omega$ (resp. $\tilde{\varphi}_\omega$) and $\tilde{\chi}'_\omega$ (resp. $\tilde{\varphi}'_\omega$) the respective primitive (resp. period) maps : these are diffeomorphisms (resp. isomorphisms). Write $\tilde{\chi}_\omega = \tilde{\chi}'_\omega \circ \psi$. Then $\tilde{\varphi}_\omega = \tilde{\varphi}'_\omega \circ T\psi$ where

$$T\psi : \Theta_{\tilde{M}} \to \Theta_{\tilde{M}'}$$

is the linear tangent map: it is an isomorphism which carries the structures from $\Theta_{\tilde{M}}$ onto $\Theta_{\tilde{M}'}$. □

Example 3.2.2 A Frobenius type structure (E, R_0, R_∞, g) on a point has a universal deformation if R_0 is regular. This result was already known by B. Malgrange [11]. One gets a Frobenius structure on the base of any universal deformation of a regular Frobenius type structure if moreover ω is *homogeneous*. This has been independently shown by B. Dubrovin [6].

4 Frobenius type structures and Laurent polynomials

We explain here, and it is the first step, how to attach a Frobenius type structure to any convenient and nondegenerate Laurent polynomial.

Until the end of this paper, f will denote a convenient and nondegenerate Laurent polynomial defined on U.

4.1 Subdiagram deformations

If f has a finite number of critical points, $\mu(f)$ will denote its global Milnor number, that is the sum of the Milnor numbers at its critical points. One attaches to f its Newton polyhedron and an increasing filtration $\mathcal{N}_\bullet$ on K, indexed by $\mathbb{Q}$ and normalized such that $f \in \mathcal{N}_1 K$ (see [9], we keep here the notations of [2]): this is the Newton filtration. This filtration induces a Newton filtration $\mathcal{N}_\bullet$ on $\Omega^n(U)$ such that $du/u \in \mathcal{N}_0\Omega^n(U)$. Define

$$\mathcal{N}_{<1}K := \cup_{\alpha<1}\mathcal{N}_\alpha K,$$

which is a finite dimensional $\mathbb{C}$-vector space, and $\nu := \dim_{\mathbb{C}} \mathcal{N}_{<1}K$. Let

$$F : U \times \mathbb{C}^r \to \mathbb{C}$$

be the deformation of f defined by

$$F(u, x) = f(u) + \sum_{i=1}^{r} x_i g_i(u),$$

the g_i's being Laurent polynomials.

Definition 4.1.1 (1) A Laurent polynomial g is *subdiagram* if $g \in \mathcal{N}_{<1}K$.

(2) F is a *subdiagram* deformation of f if the Laurent polynomials g_i, $i = 1, \ldots, r$, are subdiagram.

(3) The subdiagram deformation F is *injective* if the g_i's are $\mathbb{C}$-linearly independent, *maximal* if it is injective and if $r = \nu$ and *surjective* if $(g_1, \ldots, g_r)$ is a lattice in A_f, that is if every element in A_f can be written as (the class of) a polynomial in $g_1, \ldots, g_r$ with coefficients in $\mathbb{C}$.

Remark 4.1.2 Let $F_1^{\max}$ and $F_2^{\max}$ be two maximal subdiagram deformations. Then $F_1^{\max}$ is surjective if and only if $F_2^{\max}$ is so. In particular, if a maximal subdiagram deformation is surjective then any maximal subdiagram deformation will be so.

4.2 The Brieskorn lattice of a subdiagram deformation

Let F be a subdiagram deformation of f, G_0 (resp. G) the (dual) Fourier–Laplace transform of the Brieskorn lattice (resp. of the Gauss–Manin system) of F, G_0^o (resp. G^o) the one of f, see [4, Sect. 2.c]. By the very definition, one has

$$G_0^o = \frac{\Omega^n(U)[\theta]}{(\theta d - df\wedge)\Omega^{n-1}(U)[\theta]},$$

$$G_0 = \frac{\Omega^n(U)[x, \theta]}{(\theta d_u - d_u F\wedge)\Omega^{n-1}(U)[x, \theta]},$$

where the notation d_u means that the differential is taken with respect to u only,

$$G = \frac{\Omega^n(U)[x, \theta, \theta^{-1}]}{(\theta d_u - d_u F\wedge)\Omega^{n-1}(U)[x, \theta, \theta^{-1}]}$$

and

$$G^o = \frac{\Omega^n(U)[\theta, \theta^{-1}]}{(\theta d - df\wedge)\Omega^{n-1}(U)[\theta, \theta^{-1}]}.$$

G_0 is a $\mathbb{C}[x,\theta]$-module and G_0^o is a $\mathbb{C}[\theta]$-module. One defines a connection ∇ on G putting, for $\omega \in \Omega^n(U)[x]$,

$$\theta^2\nabla_\theta(\omega\theta^p) = F\omega\theta^p + p\omega\theta^{p+1}$$

and

$$\nabla_{\partial_{x_j}}(\omega\theta^p) = \partial_{x_j}(\omega)\theta^p - \frac{\partial F}{\partial x_j}\omega\theta^{p-1}.$$

Notice that G_0 is stable under $\theta^2\nabla_\theta$. One defines in the same way the Brieskorn lattice G_0^a and the Gauss–Manin system G^a of $F_a := F(.,a)$.

Recall that the spectrum of (G_0^o, G^o) is the set of the $\mu(f)$ rational numbers $(\alpha_1,\dots,\alpha_\mu)$ such that

$$\sharp(i|\alpha_i = \alpha) = \dim_{\mathbb{C}} \frac{\mathcal{N}_\alpha\Omega^n(U)}{(df\wedge\Omega^{n-1}(U))\cap\mathcal{N}_\alpha\Omega^n(U) + \mathcal{N}_{<\alpha}\Omega^n(U)}.$$

Theorem 4.2.1 (1) *$\mu(f) < +\infty$ and G_0^o is a free $\mathbb{C}[\theta]$-module of rank $\mu(f)$.*
(2) *The Brieskorn lattice G_0 of any subdiagram deformation F of f is free, of rank $\mu(f)$, over $\mathbb{C}[x,\theta]$.*
(3) *Let F be a subdiagram deformation of f. For any value a of the parameter, one has $\mu(F_a) = \mu(f)$ and the spectrum of (G_0^a, G^a) is equal to the one of (G_0^o, G^o).*

Proof From [9], one gets $\mu(f) < +\infty$ because f is convenient and nondegenerate. The remaining assertions of (1) and (2) follow from the division theorem of Kouchnirenko, as stated in [4, Lemma 4.6]: see [4, Remark 4.8] for (1) and [2, Proposition 2.2.1] for (2). Let us show (3): if f is convenient and nondegenerate, F_a is so and the Newton polyhedra of f and F_a are the same : thus, the first assertion follows also from [9]. If $\sum_i a_i u_i \frac{\partial f}{\partial u_i} \in \mathcal{N}_\alpha K$ one may assume, because of the division theorem quoted above, that $a_i \in \mathcal{N}_{\alpha-1}K$. Since the g_j's are subdiagram, one gets $u_i\frac{\partial g_j}{\partial u_i} \in \mathcal{N}_{<1}K$. It follows that

$$(df\wedge\Omega^{n-1}(U))\cap\mathcal{N}_\alpha + \mathcal{N}_{<\alpha} = (dF_a\wedge\Omega^{n-1}(U))\cap\mathcal{N}_\alpha + \mathcal{N}_{<\alpha}.$$

This gives the second assertion. □

4.3 The canonical Frobenius type structure of a subdiagram deformation

Assume, and it is the starting point, that one has solved the Birkhoff problem for G_0^o, that is that one has found a basis $\varepsilon^o = (\varepsilon_1^o,\dots,\varepsilon_\mu^o)$ (we put here $\mu = \mu(f)$) of G_0^o over $\mathbb{C}[\theta]$, adapted to the microlocal Poincare duality S^o (see [15], [5, p. 9] and also [2, Paragraph 3.3]), in which the matrix of the Gauss–Manin connection takes the form

$$-(\tau A_0^o + A_\infty)\frac{d\tau}{\tau}$$

(we put $\tau := \theta^{-1}$). This means that one can extend G_0^o to a trivial bundle on $\mathbb{P}^1$ equipped with a meromorphic connection with logarithmic poles along $\tau = 0$ and poles of order 1 along $\tau = \infty$. One gets, using Sect. 2.2.2, a Frobenius type structure $(E^o, R_0^o, R_\infty, g^o)$ on a point where

- $E^o = G_0^o/\theta G_0^o = \Omega^n(U)/df\wedge\Omega^{n-1}(U)$,
- R_0^o (resp. R_∞) is the endomorphism of E^o whose matrix is A_0^o (resp. A_∞) in the basis induced by ε^o.

It follows from Sect. 4.2 that R_0^o is the multiplication by f on E^o.

In this paper, we will always consider the canonical solution of the Birkhoff problem given by M. Saito's method [15], [4, Appendix B], [3, Sect. 6]. To any convenient and nondegenerate Laurent polynomial f, one attaches in this way a *canonical* Frobenius type structure on a point. The endomorphism R_∞ is in particular semisimple and its eigenvalues run through the spectrum of (G_0^o, G^o). The basis ε^o is homogeneous, that is $R_\infty(\varepsilon_i^o) = \alpha_i \varepsilon_i^o$ for all i, and we order ε^o such that

$$\alpha_1 \leq \cdots \leq \alpha_\mu.$$

Since f is a convenient and nondegenerate Laurent polynomial, one has moreover $\varepsilon_1^o = [\frac{du}{u}]$ where [ ] denotes the class in G_0^o, $\alpha_1 = 0 < \alpha_2$ (the multiplicity of α_1 in the spectrum is equal to 1) and $\alpha_\mu = n > \alpha_{\mu-1}$ (see [4, 4.d]).

Theorem 4.3.1 *Let F be a subdiagram deformation of f and*

$$E = G_0/\theta G_0 = \Omega^n(U)[x]/d_u F \wedge \Omega^{n-1}(U)[x].$$

The construction in Sect. 2.2.2 attaches to the canonical solution of the Birkhoff problem for G_0^o a unique Frobenius type structure

$$\mathbb{F}_o = (\mathbb{A}^r, E, \bigtriangledown, R_0, R_\infty, \Phi, g)$$

such that

$$i_{\{0\}}^* \mathbb{F}_o = (E^o, R_0^o, R_\infty, g^o).$$

Moreover, for any value a of the parameter, one has

$$i_{\{a\}}^* \mathbb{F}_o = (E^a, R_0^a, R_\infty, g^a),$$

$(E^a, R_0^a, R_\infty, g^a)$ denoting the canonical Frobenius type structure attached to $F_a := F(., a)$.

Proof It follows from [2, Corollary 3.1.3] that there exists a basis $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\mu)$ of G_0 over $\mathbb{C}[x, \theta]$ such that:

1. the matrix of the connection ∇ in this basis takes the form

$$-(\tau A_0(x) + A_\infty)\frac{d\tau}{\tau} + \tau C(x),$$

   where $C(x) = \sum_{i=1}^r C^{(i)}(x)dx_i$. The matrix $A_0(x)$ represents the multiplication by F on $G_0/\tau^{-1}G_0$ in the basis induced by ε. Its coefficients belong to $\mathbb{C}[x]$. The matrix $C^{(i)}$ represents the multiplication by $-g_i$ on $G_0/\tau^{-1}G_0$. Its coefficients belong also to $\mathbb{C}[x]$. Last, the matrix A_∞ is constant.
2. The restriction of ε to the zero value of the parameters is equal to ε^o, the canonical solution of the Birkhoff problem for G_0^o.

The unicity of such a basis is classical (see [11] or [12, p. 209]). Now one gets the desired Frobenius type structure $\mathbb{F}_o$ using the results of Sect. 2.2.2. The construction in [2] shows also that the restriction of the solution ε to any value a of the parameter is the canonical solution of the Birkhoff problem for G_0^a. This gives the last assertion. □

Definition 4.3.2 We will say that the Frobenius type structure $\mathbb{F}_o$ constructed in Theorem 4.3.1 is the *canonical Frobenius type structure* attached to the subdiagram deformation F.

In the notation $\mathbb{F}_o$, the index $_o$ recalls the initial data (that is, f). More generally, one attaches in this way a Frobenius type structure to any (not necessarily the canonical one) solution of the Birkhoff problem for G_0^o.

4.4 Comparison of the canonical Frobenius type structures after a change of initial condition

Let F be a subdiagram deformation of f and $(E^a, R_0^a, R_\infty, g^a)$ be the canonical Frobenius type structure on a point attached to $F_a = F(., a)$. Let us also consider the subdiagram deformation of F_a defined by

$$(u, x) \mapsto F(u, x + a).$$

Theorem 4.3.1 attaches to F_a a Frobenius type structure on $\mathbb{A}^r$

$$\mathbb{F}_a = (\mathbb{A}^r, E, \bigtriangledown, R_0, R_\infty, \Phi, g),$$

where

$$E := \frac{\Omega^n(U)[x]}{d_u F(u, x + a) \wedge \Omega^{n-1}(U)[x]}$$

and such that

$$i^*_{\{0\}}\mathbb{F}_a = (E^a, R_0^a, R_\infty, g^a).$$

Let ρ_a be the map defined by $\rho_a(x) = x + a$.

Proposition 4.4.1 *For any a one has $\mathbb{F}_a = \rho_a^* \mathbb{F}_o$.*

Proof Follows from the unicity given by Theorem 4.3.1. □

In other words, the matrices attached by Lemma 2.1.2 to the Frobenius type structures involved are related by a translation.

4.5 Comparison of the canonical Frobenius type structures attached to two different subdiagram deformations

We now compare the canonical Frobenius type structures attached to two different subdiagram deformations.

Proposition 4.5.1 (1) *Let $F^{\max}$ and $G^{\max}$ be two subdiagram maximal deformations of f, $\mathbb{F}_o^{\max}$ and $\mathbb{G}_o^{\max}$ the canonical Frobenius type structures attached to $F^{\max}$ and $G^{\max}$ by Theorem 4.3.1. Then $\mathbb{F}_o^{\max}$ and $\mathbb{G}_o^{\max}$ are isomorphic.*

(2) *Let $\mathbb{F}_o$ be the canonical Frobenius type structure attached to an injective subdiagram deformation F, $\mathbb{G}_o^{\max}$ the canonical Frobenius type structure attached to a maximal subdiagram deformation $G^{\max}$. Then $\mathbb{F}_o$ is induced by $\mathbb{G}_o^{\max}$: there exists a map $\Psi : \mathbb{A}^r \to \mathbb{A}^\nu$ such that $\mathbb{F}_o = \Psi^* \mathbb{G}_o^{\max}$.*

Proof Write

$$F^{\max}(u, x) = f(u) + \sum_{i=1}^{\nu} x_i g_i$$

and

$$G^{\max}(u, x) = f(u) + \sum_{i=1}^{\nu} x_i g_i'.$$

Since $F^{\max}$ and $G^{\max}$ are maximal, (g_i) and (g_i') are two bases of $\mathcal{N}_{<1}K$ and there exists independent linear forms $L_1, \ldots, L_\nu$ such that

$$G^{\max}(u, x) = f(u) + \sum_{i=1}^{\nu} L_i(x_1, \ldots, x_\nu) g_i.$$

Define the map Φ by

$$\Phi(x_1, \ldots, x_\nu) = (L_1(x_1, \ldots, x_\nu), \ldots, L_\nu(x_1, \ldots, x_\nu)).$$

Then $\mathbb{G}_o^{\max} = \Phi^* \mathbb{F}_o^{\max}$. This shows (1) and (2) follows from (1). □

4.6 Good subdiagram deformations

We define in this section a class of distinguished subdiagram deformations. We will use these deformations in order to construct global deformations of the canonical Frobenius type structures along the subdiagram polynomials (see Sect. 6.2). If F is a subdiagram deformation of f, let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\mu)$ be the canonical solution of the Birkhoff problem for G_0 given by the proof of Theorem 4.3.1. Recall that $R_\infty(\varepsilon_i) = \alpha_i \varepsilon_i$ for some rational numbers α_i. We *order* ε as in the beginning of Sect. 4.3. Let $\mathbb{F}_o$ be the canonical Frobenius type structure attached to F: we thus have a map

$$\Phi : E \to \Omega^1(\mathbb{A}^r) \otimes E.$$

Write $\Phi = \sum_i \Phi^{(i)} dx_i$. By definition, the $\Phi^{(i)}$'s are endomorphisms of E.

Definition 4.6.1 We will say that a subdiagram deformation F is *good* if F is injective and if, for all i,

$$-\Phi^{(i)}(\varepsilon_1) = \varepsilon_i + \sum_{j<i} a_i^j(x) \varepsilon_j,$$

where $a_i^j \in \mathbb{C}[x]$.

Proposition 4.6.2 *There exist good (resp. good and maximal) subdiagram deformations.*

We will denote a good (resp. a good and maximal) subdiagram deformation by F^{good} (resp. $F^{\mathrm{good,max}}$).

Proof It is enough to work on the fiber above 0: indeed, if $-\Phi^{(i)}(\varepsilon_1^o) = \varepsilon_i^o$ for all i one gets

$$-\Phi^{(i)}(\varepsilon_1) = \varepsilon_i + \sum_{j<i} a_i^j(x) \varepsilon_j$$

because, the deformation being subdiagram, the principal parts are constant (see [2]). Define, if $R_\infty(\varepsilon_i^o) = \alpha_i \varepsilon_i^o$,

$$\square_\alpha := \sum_{\alpha_i \leq \alpha} \mathbb{C} \varepsilon_i^o.$$

By construction, one has (see [4, Appendix B] or [3, Paragraph 6])

$$\frac{\square_\alpha}{\square_{<\alpha}} = gr_\alpha^{\mathcal{N}} E^o,$$

where $E^o = \Omega^n(U)/df \wedge \Omega^{n-1}(U)$ and $\mathcal{N}_\bullet$ is the Newton filtration induced on E^o. If $\alpha < 1$, it follows from [4, Lemma 4.6] that

$$gr_\alpha^{\mathcal{N}} E^o = gr_\alpha^{\mathcal{N}} \Omega^n(U).$$

Since $\mathcal{N}_{<0}\Omega^n(U) = \square_{<0} = 0$, one deduces that

$$\mathcal{N}_\alpha \Omega^n(U) = \square_\alpha$$

for all $\alpha < 1$. This shows two things : first, one has $\alpha_i < 1$ for all $i \in \{1, \dots, \nu\}$ and second, given ε_i^o such that $\alpha_i < 1$, there exists a unique subdiagram Laurent polynomial g_i such that

$$\left[g_i \frac{du}{u} \right] = \varepsilon_i^o.$$

Then, for $r \leq \nu$,

$$F^{\text{good}}(u, x) = f(u) + \sum_{i=1}^{r} x_i g_i$$

is clearly injective and is a good subdiagram deformation because $\Phi^{(i)}$ is the multiplication by $-g_i$ and $\varepsilon_1^o = [du/u]$. The subdiagram deformation

$$F^{\text{good,max}}(u, x) = f(u) + \sum_{i=1}^{\nu} x_i g_i$$

is good and maximal. $\square$

Let $\mathbb{F}_o^{\text{good}}$ (resp. $\mathbb{F}_o^{\text{good,max}}$) be the canonical Frobenius type structure attached to the good (resp. to the good and maximal) subdiagram deformation F^{good} (resp. $F^{\text{good,max}}$).

Lemma 4.6.3 (1) *Assume that there exist subdiagram Laurent polynomials $g_1, \dots, g_r$ such that $(g_1, \dots, g_r)$ is a lattice in A_f and let $F^{\text{good,max}}$ be a good and maximal subdiagram deformation. Then $F^{\text{good,max}}$ is surjective.*

(2) *$\mathbb{F}_o^{\text{good,max}}$ is isomorphic to the canonical Frobenius type structure attached to any maximal subdiagram deformation and it induces the canonical Frobenius type structure attached to any injective subdiagram deformation.*

Proof (1) Follows from Remark 4.1.2 and (2) follows from Proposition 4.5.1 because $F^{\text{good,max}}$ is a maximal subdiagram deformation. $\square$

5 Pre-primitive forms of a canonical Frobenius type structure

Let f be a convenient and nondegenerate Laurent polynomial,

$$F(u, x) = f(u) + \sum_{i=1}^{r} x_i g_i$$

be a subdiagram deformation of f and $\mathbb{F}_o = (\mathbb{A}^r, E, \bigtriangledown, R_0, R_\infty, \Phi, g)$ be the canonical Frobenius type structure on $\mathbb{A}^r$ attached to F.

5.1 The section ω

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\mu)$ be the ordered solution of the Birkhoff problem for G_0, as in Sect. 4.6.

Proposition 5.1.1 *One has $\varepsilon_1 = [\frac{du}{u}]$ where [ ] denotes the class in G_0. In particular, the class of $\frac{du}{u}$ in E is ∇-flat and homogeneous, i.e. an eigenvector of R_∞.*

Proof Let $V_\bullet$ be the Malgrange-Kashiwara filtration along $\tau = 0$ of the Gauss–Manin system G of the subdiagram deformation F, $V_\bullet G_0$ its trace on G_0. In the convenient and nondegenerate case, this filtration is equal to the Newton filtration (up to a shift, see [2, Proposition 2.3.3]). It follows from [2, Proposition 2.3.1] that $V_{\alpha_1} G_0$ is a free $\mathbb{C}[x]$-module and, from [3, Proposition 7.0.2], that every basis of $V_{\alpha_1} G_0$ is a part of a solution of the Birkhoff problem for G_0. Now, $V_{\alpha_1} G_0$ is of rank 1 over $\mathbb{C}[x]$ (for all a the $\mathbb{C}$-vector space $V_{\alpha_1} G_0^a$ is 1-dimensional because F_a is a convenient and nondegenerate Laurent polynomial, see [4, 4.d]), ε_1 is a basis of it and $\frac{du}{u} \in V_{\alpha_1} G_0$. Notice that, once again because F_a is a convenient and nondegenerate Laurent polynomial, ε_1^a is equal to the class of the form du/u in G_0^a for all a [4, 4.d]. If $[du/u] = p(x)\varepsilon_1$ in G_0, we deduce from this that $p(x) = 1$ for all x. □

Notation 5.1.2 Until the end of this paper, ω will denote the class of $\frac{du}{u}$ in E.

5.2 Conditions (IC) and (GC) for ω^o

Choose an origin, say 0. We have

$$E^o = E/(x)E = \Omega^n(U)/df \wedge \Omega^{n-1}(U)$$

and ω^o denotes the class of $\frac{du}{u}$ in E^o. Conditions (IC) and (GC) for ω^o are defined in 3.1.7.

Lemma 5.2.1 (1) *ω^o satisfies (IC) if and only if the classes of $g_1, \ldots, g_r$ are linearly independent in A_f.*
(2) *ω^o satisfies (GC) if and only if every element of A_f can be written as (the class of) a polynomial in $g_1, \ldots, g_r, f$ with coefficients in $\mathbb{C}$.*

Proof By definition (see Sect. 4.2), one has $R_0^o(\omega^o) = [f\frac{du}{u}]$ and $-\Phi^o_{\partial_{x_i}}(\omega) = [g_i \frac{du}{u}]$ where [ ] denotes the class in E^o. □

The following proposition justifies Definition 4.1.1:

Proposition 5.2.2 (1) *Assume that the deformation F is injective. Then ω^o satisfies (IC).*
(2) *Assume that the deformation F is surjective. Then ω^o satisfies (GC).*

Proof Let us show (1): it is enough to show that the classes of $g_1, \ldots, g_r$ in A_f are linearly independent. But this follows from the conditions $g_j \in \mathcal{N}_{\alpha_j} K$ with $\alpha_j < 1$ for all j: indeed, assume that there exist complex numbers $\alpha_1, \ldots, \alpha_r$ such that

$$\sum_{j=1}^{r} \alpha_j g_j = \sum_{i=1}^{n} b_i u_i \frac{\partial f}{\partial u_i}.$$

One can choose, using [4, Lemma 4.6], the b_i's such that $b_i \in \mathcal{N}_{\alpha-1} K$ where $\alpha := max_j \alpha_j$. We then get $b_i = 0$ for all i because $\alpha < 1$. Moreover, the g_j's are linearly independent in K because the deformation F is injective: this shows that $\alpha_i = 0$ for all i. (2) is clear. □

Example 5.2.3 Assume that the injective deformation F contains the monomials

$$u_1, \ldots, u_n, u_1^{-1}, \ldots, u_n^{-1}.$$

Then ω^o satisfies (IC) and (GC). Notice that, often, the monomials $1/u_1, \ldots, 1/u_n$ are equal, in A_f, to a (positive) power of the monomials $u_1, \ldots, u_n$: in this case, the condition "*F contains the monomials $u_1, \ldots, u_n$*" is enough to get the condition (GC) for ω^o.

Lemma 5.2.4 *Assume that the deformation F is injective. Then ω^a satisfies (IC) for any choice of origin a.*

Proof It is enough to show that the classes of $g_1, \ldots, g_r$ in A_{F_a} are linearly independent. But one can repeat the proof of the previous proposition, because F_a is convenient and nondegenerate and because the Newton polyhedra (and hence the Newton filtrations) of f and F_a are the same. □

5.3 The canonical pre-primitive form

Le $\mathbb{F}_o^{\mathrm{an}}$ be the analytization of the Frobenius type structure $\mathbb{F}_o$ (see Remark 3.1.8), $\mathbb{F}_{o,0}^{\mathrm{an}}$ its germ at 0.

Proposition 5.3.1 (1) *Assume that the subdiagram deformation F is injective and surjective. Then ω^{an} is a pre-primitive section of $\mathbb{F}_{o,0}^{\mathrm{an}}$.*

(2) *Assume that the subdiagram deformation F is injective. Then, the section ω of $\mathbb{F}_o$ satisfies $(IC)^{gl}$. If moreover F contains the monomials $u_1, \ldots, u_n, u_1^{-1}, \ldots, u_n^{-1}$ then ω satisfies also $(GC)^{gl}$ and ω^{an} is a pre-primitive section of the Frobenius type structure $\mathbb{F}_o^{\mathrm{an}}$ for any choice of the origin in $\mathbb{C}^r$.*

Proof (1) follows from Proposition 5.2.2. A section of the kernel of the period map φ_ω is given by a finite number of polynomials that vanishes everywhere by Lemma 5.2.4. This shows the first assertion of (2). With the given assumption, ω satisfies of course $(GC)^{gl}$ and the results about ω^{an} are then clear. □

6 Deformations and universal deformations of the canonical Frobenius type structure

We keep here the situation and the notations of Sect. 5.

6.1 Deformations of the canonical Frobenius type structure

Let $C(x)$, $B_0(x)$ and B_∞ be the matrices attached to $\mathbb{F}_o$ by Lemma 2.1.2. Recall the conditions $(GC)^{gl}$ and $(IC)^{gl}$ for ω, given in Definition 3.1.7.

Lemma 6.1.1 *Assume that ω satisfies $(GC)^{gl}$. Let $f_{11}, \ldots, f_{\mu 1}$ be elements of $\mathbb{C}[x]\{y\}$ (resp. $\mathcal{O}(\mathbb{C}^r)\{y\}$), $y \in \mathbb{C}$, such that $f_{i1}(x, 0) = 0$ for $i = 1, \ldots, \mu$. Then there exists a unique t-uple of matrices*

$$(C(x, y), B_0(x, y), B_\infty)$$

such that

(1) *the coefficients of $C(x, y)$ and $B_0(x, y)$ belong to $\mathbb{C}[x]\{y\}$ (resp. $\mathcal{O}(\mathbb{C}^r)\{y\}$),*

(2) *$C(x,0)=C(x)$, $B_0(x,0)=B_0(x)$ and $\frac{\partial f_{i1}}{\partial y}(x,y)=D_{i1}(x,y)$ if*

$$C(x,y)=\sum_{i=1}^{r}C^{(i)}(x,y)dx_i+D(x,y)dy,$$

(3) *the relations of Lemma 2.1.2 are satisfied.*

Proof See [8, Theorem 2.5]. It remains to show that the coefficients of $C(x,y)$ and $B_0(x,y)$ belong to $\mathbb{C}[x]\{y\}$ (resp. $\mathcal{O}(\mathbb{C}^r)\{y\}$), but this follows from the fact that the coefficients of $C(x)$ and $B_0(x)$ belong to $\mathbb{C}[x]$ by Theorem 4.3.1 and from the condition $(GC)^{gl}$. □

Example 6.1.2 Assume that $f_{11}(x,y)=y$ and $f_{i1}(x,y)=0$ for $i=2,\dots,\mu$. Lemma 2.1.2 gives

$$C_{j1}^{(i)}(x,y)=C_{j1}^{(i)}(x)$$

for all i and for all j, $D_{11}(x,y)=1$ and $D_{j1}(x,y)=0$ if $j\neq 1$.

By induction, one shows that Lemma 6.1.1 remains true if $y=(y_1,\dots,y_\ell)\in\mathbb{C}^\ell$.

Corollary 6.1.3 *Assume that ω satisfies $(GC)^{gl}$. Then,*

(1) *for any choice of functions*

$$f_{11},\dots,f_{\mu 1}\in\mathcal{O}(\mathbb{C}^r)\{y_1,\dots,y_\ell\}$$

such that $f_{i1}(x,0)=0$ there exists a unique *deformation*

$$\tilde{\mathbb{F}}_o^{\mathrm{an}}=(\mathbb{C}^r\times(\mathbb{C}^\ell,0),\tilde{E},\tilde{\bigtriangledown},\tilde{R}_0,\tilde{R}_\infty,\tilde{\Phi},\tilde{g})$$

on $\mathbb{C}^r\times(\mathbb{C}^\ell,0)$ of the canonical Frobenius type structure $\mathbb{F}_o^{\mathrm{an}}$.

(2) *Any deformation of $\mathbb{F}_o^{\mathrm{an}}$ on $\mathbb{C}^r\times(\mathbb{C}^{\ell'},0)$ can be obtained as in (1).*

Proof For (1), it remains to show the assertion on the metric $\tilde{g}$: $\tilde{g}$ is the unique $\tilde{\bigtriangledown}$-flat metric on $\tilde{E}$ extending g. Starting with a basis adapted to g, and keeping the notations of Lemma 6.1.1, it suffices to show that if the initial data are symmetric, then the matrices $C(x,y)$ and $B_0(x,y)$ are so : one can argue by induction as in the proof of Lemma 6.1.1 (see [10, Corollary 1.17], [8, Lemma 3.2] and also [2, Paragraph 3.3]). Let us show (2): let $\check{\mathbb{F}}_o^{\mathrm{an}}$ be a deformation of $\mathbb{F}_o^{\mathrm{an}}$ on $\mathbb{C}^r\times(\mathbb{C}^{\ell'},0)$,

$$\check{\chi}_{\omega^{\mathrm{an}}}:\mathbb{C}^r\times(\mathbb{C}^{\ell'},0)\to\mathbb{C}^\mu, \tag{9}$$

$$(x,y)\mapsto(\Gamma_{11}(x,y),\dots,\Gamma_{\mu 1}(x,y)) \tag{10}$$

its primitive map (attached to the flat extension of ω^{an}). One puts $f_{i1}(x,y)=\Gamma_{i1}(x,y)-\Gamma_{i1}(x,0)$. □

Remark 6.1.4 (Local deformations of the canonical Frobenius type structure) Assume that ω^o satisfies (GC) for the origin 0. One gets in the same way deformations $\tilde{\mathbb{F}}_{o,0}^{\mathrm{an}}$ of $\mathbb{F}_{o,0}^{\mathrm{an}}:=\mathbb{C}\{x\}\otimes\mathbb{F}_o$. The functions f_{i1} now belong to $\mathbb{C}\{x,y\}$ and the coefficients of the matrices involved are holomorphic. This is the setting of [8]. If moreover ω^o satisfies (IC), one can choose the f_{i1}'s such that $\tilde{\chi}_{\omega^{\mathrm{an}}}$, the primitive map attached to $\tilde{\mathbb{F}}_{o,0}^{\mathrm{an}}$, is (at least locally) invertible: $\tilde{\mathbb{F}}_{o,0}^{\mathrm{an}}$ is then a universal deformation of $\mathbb{F}_{o,0}^{\mathrm{an}}$. This is precisely what gives [8, p. 123]. In particular, and because of Proposition 5.2.2, the canonical Frobenius type structure attached to an injective and surjective subdiagram deformation of f has a universal deformation $\tilde{\mathbb{F}}_{o,0}^{\mathrm{an}}$. In this situation, a deformation $\check{\mathbb{F}}_{o,0}^{\mathrm{an}}$ of $\mathbb{F}_{o,0}^{\mathrm{an}}$ is induced from $\tilde{\mathbb{F}}_{o,0}^{\mathrm{an}}$ by the map $\psi=\tilde{\chi}_{\omega^{\mathrm{an}}}^{-1}\circ\check{\chi}_{\omega^{\mathrm{an}}}$ where $\check{\chi}_{\omega^{\mathrm{an}}}$ is the primitive map attached to $\check{\mathbb{F}}_{o,0}^{\mathrm{an}}$.

Let $a \in \mathbb{C}^r$ and ρ_a be the map defined by $\rho_a(x, y) = (x + a, y)$. From Proposition 4.4.1 we get

Corollary 6.1.5 *Assume that ω satisfies condition $(GC)^{gl}$. Let $\tilde{\mathbb{F}}_o^{\mathrm{an}}$ be the deformation of $\mathbb{F}_o^{\mathrm{an}}$ given by Corollary 6.1.3 for a choice of functions f_{i1}. Then $\rho_a^*\tilde{\mathbb{F}}_o^{\mathrm{an}}$ is the deformation of the Frobenius type structure $\mathbb{F}_a^{\mathrm{an}}$ given by Corollary 6.1.3 for the functions $f_{i1} \circ \rho_a$.*

6.2 Semi-global universal deformations of the canonical Frobenius type structure

We globalize here the results of Remark 6.1.4 along $\mathbb{C}^r$ (i.e. along the subdiagram monomials). We give first the analog of definition 2.3.1, (3):

Definition 6.2.1 Let $\tilde{\mathbb{F}}_o^{\mathrm{an}}$ be a deformation of $\mathbb{F}_o^{\mathrm{an}}$ on $\mathbb{C}^r \times (\mathbb{C}^\ell, 0)$ as in Corollary 6.1.3. We say that $\tilde{\mathbb{F}}_o^{\mathrm{an}}$ is a *semi-global universal* deformation of $\mathbb{F}_o^{\mathrm{an}}$ if, for any other deformation $\tilde{\mathbb{F}}_o^{\mathrm{an}'}$ on $\mathbb{C}^r \times (\mathbb{C}^{\ell'}, 0)$ of $\mathbb{F}_o^{\mathrm{an}}$, there exists a unique map

$$\Psi : \mathbb{C}^r \times (\mathbb{C}^{\ell'}, 0) \to \mathbb{C}^r \times (\mathbb{C}^\ell, 0),$$

inducing the identity on $\mathbb{C}^r$ and such that $\Psi^*\tilde{\mathbb{F}}_o^{\mathrm{an}} = \tilde{\mathbb{F}}_o^{\mathrm{an}'}$.

We show first that such semi-global universal deformations exist if F is a good subdiagram deformation (see Sect. 4.6).

Lemma 6.2.2 *Let $\mathbb{F}_o^{\mathrm{good}}$ be the canonical Frobenius type structure attached to a good subdiagram deformation of f. Then*:

(1) *the primitive map χ_ω attached to $\mathbb{F}_o^{\mathrm{good}}$ takes the form*

$$\chi_\omega(x_1, \ldots, x_r) = (-x_1 + G_1(x_2, \ldots, x_r), -x_2 + G_2(x_3, \ldots, x_r), \ldots,\\ -x_{r-1} + G_{r-1}(x_r), -x_r, 0, \ldots, 0),$$

where G_1, G_2,..., G_{r-1} are suitable polynomial functions.

(2) *Assume moreover that ω satisfies $(GC)^{gl}$. Choose $f_{i1}(x, y) = 0$ for $i = 1, \ldots, r$, $f_{i1}(x, y) = y_{i-r}$ for $i = r + 1, \ldots, \mu$ and let $\tilde{\mathbb{F}}_o^{\mathrm{good,an}}$ be the deformation of $\mathbb{F}_o^{\mathrm{good,an}}$ given by Corollary 6.1.3. Its primitive map*

$$\tilde{\chi}_{\omega^{\mathrm{an}}} : \mathbb{C}^r \times (\mathbb{C}^{\mu-r}, 0) \to \mathbb{C}^r \times (\mathbb{C}^{\mu-r}, 0)$$

takes the form

$$\tilde{\chi}_{\omega^{\mathrm{an}}}(x_1, \ldots, x_r, y_1, \ldots, y_{\mu-r}) = (-x_1 + G_1(x_2, \ldots, x_r), \ldots, -x_{r-1}\\ +G_{r-1}(x_r), -x_r, y_1, \ldots, y_{\mu-r})$$

and $\tilde{\mathbb{F}}_o^{\mathrm{good,an}}$ is a semi-global universal deformation of $\mathbb{F}_o^{\mathrm{good,an}}$.

Proof (1) By definition of the good subdiagram deformations, we have, in G_0,

$$-\Phi^{(i)}(\omega) = \varepsilon_i + \sum_{j<i} a_i^j(x)\varepsilon_j$$

for all i, with $a_i^j \in \mathbb{C}[x]$. Let Γ_{j1} be such that

$$d\Gamma_{j1}(x) = \sum_{i=1}^{r} C_{j1}^{(i)}(x)dx_i$$

with the initial data $\Gamma_{j1}(0) = 0$. One has $d\Gamma_{j1}(x) = 0$ for $j > r$ hence $\Gamma_{j1}(x) = 0$ for $j > r$. In the same way, one gets $d\Gamma_{r1}(x) = -dx_r$ and

$$d\Gamma_{j1}(x) = -dx_j + \sum_{i=j+1}^{r} C_{j1}^{(i)}(x)dx_i$$

for $j = 1, \ldots, r-1$. The result follows. Now (2) follows from (1) and Example 6.1.2. One gets the universality as in Remark 6.1.4 (notice that $\tilde{\chi}_{\omega^{\mathrm{an}}}^{-1}$ is also polynomial in x). □

Corollary 6.2.3 *The canonical Frobenius type structure attached to a* maximal *subdiagram deformation has semi-global universal deformations, if ω satisfies $(GC)^{gl}$.*

Proof Apply the previous Lemma to a good and maximal subdiagram deformation, which exists by Proposition 4.6.2, and use Lemma 4.6.3 (2). □

Finally, using Corollary 6.1.5, we get

Corollary 6.2.4 *Let F be a maximal subdiagram deformation of f and assume that ω satisfies $(GC)^{gl}$. For any $a \in \mathbb{C}^r$, the Frobenius type structure $\mathbb{F}_a^{\mathrm{an}}$ has a semi-global universal deformation of $\tilde{\mathbb{F}}_a^{\mathrm{an}}$ satisfying*

$$\tilde{\mathbb{F}}_a^{\mathrm{an}} = \rho_a^* \tilde{\mathbb{F}}_o^{\mathrm{an}}.$$

7 Application: construction of Frobenius manifolds

Let f be a convenient and nondegenerate Laurent polynomial, μ its global Milnor number,

$$F(u,x) = f(u) + \sum_{i=1}^{r} x_i g_i(u)$$

be a subdiagram deformation of f, $\mathbb{F}_o = (\mathbb{A}^r, E, \nabla, R_0, R_\infty, \Phi, g)$ be the canonical Frobenius type structure attached to F by Theorem 4.3.1 and $\mathbb{F}_o^{\mathrm{an}}$ its analytization. Let ω be the class of $\frac{du}{u}$ in E.

7.1 Local setting

We work in this section with punctual germs. Let $\mathbb{F}_{o,0}^{\mathrm{an}}$ be the germ of $\mathbb{F}_o^{\mathrm{an}}$ at 0. The following theorems show that one can equip $(\mathbb{C}^\mu, 0)$ with a canonical Frobenius structure: $(\mathbb{C}^\mu, 0)$ is thus a Frobenius manifold.

Theorem 7.1.1 *Assume that the subdiagram deformation F is injective and surjective. Then*:

(1) *ω^{an} is a ∇-flat and homogeneous section of E^{an}.*
(2) *ω^{an} is pre-primitive for the origin* 0.
(3) *$\mathbb{F}_{o,0}^{\mathrm{an}}$ has a universal deformation $\tilde{\mathbb{F}}_{o,0}^{\mathrm{an}}$.*
(4) *The pre-primitive section ω^{an} defines a Frobenius structure on the base of the universal deformation $\tilde{\mathbb{F}}_{o,0}^{\mathrm{an}}$. The Frobenius structures obtained in this way on the bases of any two universal deformations are isomorphic.*

Proof (1) follows from Proposition 5.1.1, (2) from Proposition 5.2.2 and (3) from (2) and Remark 6.1.4. Last, (4) follows from Theorem 3.2.1 (2) and (3). □

Let us show now that the Frobenius structures constructed in Theorem 7.1.1 do not depend on the choice of the subdiagram deformations.

Lemma 7.1.2 *Let F (resp. G) be an injective and surjective subdiagram deformation of f, $\mathbb{F}_o$ (resp. $\mathbb{G}_o$) be the canonical Frobenius type structure attached to F (resp. G). The universal deformations $\tilde{\mathbb{F}}^{\mathrm{an}}_{o,0}$ of $\mathbb{F}^{\mathrm{an}}_{o,0}$ and $\tilde{\mathbb{G}}^{\mathrm{an}}_{o,0}$ of $\mathbb{G}^{\mathrm{an}}_{o,0}$ given by Theorem 7.1.1 are isomorphic.*

Proof Extend F (resp. G) to a maximal deformation $F^{\max}$ (resp. $G^{\max}$): this is always possible because F and G are injective. It follows from Proposition 4.5.1 that the respective canonical Frobenius structures $\mathbb{F}^{\max}_o$ and $\mathbb{G}^{\max}_o$ are isomorphic. On the other hand, $\tilde{\mathbb{F}}^{\mathrm{an}}_{o,0}$ induces $\mathbb{F}^{\max,\mathrm{an}}_{o,0}$ *via* a (unique) map ψ because $\mathbb{F}^{\max,\mathrm{an}}_{o,0}$ is a deformation of $\mathbb{F}^{\mathrm{an}}_{o,0}$ and because $\tilde{\mathbb{F}}^{\mathrm{an}}_{o,0}$ is a universal deformation of $\mathbb{F}^{\mathrm{an}}_{o,0}$. One has (see Remark 6.1.4)

$$\tilde{\chi}_{\omega^{\mathrm{an}}} \circ \psi = \chi^{\max}_{\omega^{\mathrm{an}}},$$

$\tilde{\chi}_{\omega^{\mathrm{an}}}$ (resp. $\chi^{\max}_{\omega^{\mathrm{an}}}$) denoting the primitive map attached to $\tilde{\mathbb{F}}^{\mathrm{an}}_{o,0}$ (resp. $\mathbb{F}^{\max,\mathrm{an}}_{o,0}$). By universality, $\tilde{\chi}_{\omega^{\mathrm{an}}}$ is a diffeomorphism and, by Proposition 5.2.2, $\chi^{\max}_{\omega^{\mathrm{an}}}$ is an immersion. One deduces from this that ψ is an immersion. Finally, $\tilde{\mathbb{F}}^{\mathrm{an}}_{o,0}$ is a universal deformation of $\mathbb{F}^{\max,\mathrm{an}}_{o,0}$ and also a universal deformation of $\mathbb{G}^{\max,\mathrm{an}}_{o,0}$. Because $\tilde{\mathbb{G}}^{\mathrm{an}}_{o,0}$ is also a universal deformation of $\mathbb{G}^{\max,\mathrm{an}}_{o,0}$, we deduce that $\tilde{\mathbb{F}}^{\mathrm{an}}_{o,0}$ and $\tilde{\mathbb{G}}^{\mathrm{an}}_{o,0}$ are isomorphic. □

Theorem 7.1.3 *Let F and G be two injective and surjective subdiagram deformations of f, $\mathbb{F}_o$ (resp. $\mathbb{G}_o$) be the canonical Frobenius type structure attached to F (resp. G). Then*:

(1) *ω^{an} is a $\bigtriangledown$-flat and homogeneous section of the bundles associated with $\mathbb{F}_o$ and $\mathbb{G}_o$.*
(2) *ω^{an} is pre-primitive for the origin* 0.
(3) *$\mathbb{F}^{\mathrm{an}}_o$ (resp. $\mathbb{G}^{\mathrm{an}}_o$) has a universal deformation $\tilde{\mathbb{F}}^{\mathrm{an}}_o$ (resp. $\tilde{\mathbb{G}}^{\mathrm{an}}_o$). $\tilde{\mathbb{F}}^{\mathrm{an}}_o$ and $\tilde{\mathbb{G}}^{\mathrm{an}}_o$ are isomorphic.*
(4) *The Frobenius structures defined by the pre-primitive form ω^{an} according to Theorem 7.1.1 do not depend, up to isomorphism, on the choice of the subdiagram deformations F and G.*

Proof Because of the previous Theorem, it is enough to show (3) and (4): (3) follows from the Lemma above and (4) is then clear (see Theorem 3.2.1). □

This shows Theorem 1 in the introduction.

7.2 Globalization

Recall that $\mathbb{F}_a$ denotes the canonical Frobenius type structure attached to $F_a = F(.,a)$, $a \in \mathbb{C}^r$, and that ρ_a is the map defined by $\rho_a(x,y) = (x+a,y)$ for $(x,y) \in \mathbb{C}^r \times (\mathbb{C}^{\mu-r},0)$.

Theorem 7.2.1 *Assume that the subdiagram deformation F is injective.*

(1) *ω is a $\bigtriangledown$-flat, homogeneous section of E.*
(2) *Assume moreover that ω satisfies $(GC)^{gl}$. If F is a maximal subdiagram deformation then $\mathbb{F}^{\mathrm{an}}_o$ has a semi-global universal deformation $\tilde{\mathbb{F}}^{\mathrm{an}}_o$ and, for any $a \in \mathbb{C}^r$, $\mathbb{F}^{\mathrm{an}}_a$ has a semi-global universal deformation $\tilde{\mathbb{F}}^{\mathrm{an}}_a$ satisfying $\tilde{\mathbb{F}}^{\mathrm{an}}_a = \rho^*_a \tilde{\mathbb{F}}^{\mathrm{an}}_o$. The period map ${}^o\tilde{\varphi}_{\omega^{\mathrm{an}}}$ (resp. ${}^a\tilde{\varphi}_{\omega^{\mathrm{an}}}$) attached to $\tilde{\mathbb{F}}^{\mathrm{an}}_o$ (resp. $\tilde{\mathbb{F}}^{\mathrm{an}}_a$) defines a Frobenius structure on $\mathbb{C}^r \times (\mathbb{C}^{\mu-r},0)$. If $\Phi(x,y)$ is the matrix of ${}^o\tilde{\varphi}_{\omega^{\mathrm{an}}}$ (in the obvious bases) then $\Phi(x+a,y)$ is the one of ${}^a\tilde{\varphi}_{\omega^{\mathrm{an}}}$.*

Proof (1) Follows from Proposition 5.1.1 and (2) from Corollary 6.2.3, Corollary 6.2.4 and Theorem 3.2.1. □

Corollary 7.2.2 *Assume that ω satisfies $(GC)^{gl}$ for the injective subdiagram deformation F.*

(1) *The Frobenius structure on $(\mathbb{C}^{\mu}, 0)$, attached by Theorem 7.1.1 to the convenient and nondegenerate Laurent polynomial $F_a = F(., a)$, $a \in \mathbb{C}^r$, is obtained by an analytic continuation of the one attached to f.*
(2) *For* any *injective subdiagram deformation G of f, the Frobenius structure on $(\mathbb{C}^{\mu}, 0)$, attached by Theorem 7.1.1 to the convenient and nondegenerate Laurent polynomial $G_a = G(., a)$, $a \in \mathbb{C}^r$, is obtained by an analytic continuation of the one attached to f.*

Proof Because of Theorem 7.1.3 one may assume that F is a maximal subdiagram deformation. Theorem 7.2.1 shows that ω^{an} defines a Frobenius structure on $\mathbb{C}^r \times (\mathbb{C}^{\mu-r}, 0)$ whose germ at $(0, 0) \in \mathbb{C}^r \times \mathbb{C}^{\mu-r}$ is isomorphic to the Frobenius structure given by Theorem 7.1.1 because the germ at 0 of the semi-global universal deformation $\tilde{\mathbb{F}}_o^{\mathrm{an}}$ considered in Theorem 7.2.1 is isomorphic to the universal deformation given by Theorem 7.1.1 (3) (the notation $\tilde{\mathbb{F}}_{o,0}^{\mathrm{an}}$ for both is then relevant). Now, if $a \in \mathbb{C}^r$, the same process gives also a Frobenius structure on $(\mathbb{C}^r, a) \times (\mathbb{C}^{\mu-r}, 0)$, which thus can be seen as an analytic continuation from $(0, 0)$ to $(a, 0)$ of the former one. The last assertion of Theorem 7.2.1 shows that this structure gives the one obtained on $(\mathbb{C}^{\mu}, 0)$ starting from F_a. This shows (1). Let us show (2): let G be any injective subdiagram deformation of f. Without loss of generality, one may assume that G is maximal. It follows from Proposition 4.5.1 that the canonical Frobenius type structures attached to G and F (say, $\mathbb{G}_o$ and $\mathbb{F}_o$) satisfy $\mathbb{G}_o = \Phi^*\mathbb{F}_o$ where Φ is an isomorphism. Thus, for any $a \in \mathbb{C}^r$, $\mathbb{G}_a = \Psi^*\mathbb{F}_a$ where Ψ is also an isomorphism by Proposition 4.4.1 and (2) follows from (1). □

This shows Theorem 2 in the introduction.

8 By way of conclusion

Let f be a convenient and nondegenerate Laurent polynomial. Suppose first that the multiplication by f on A_f is semisimple and regular (this case occurs in particular if the critical values of f are distinct): after [6], one can attach to f a canonical Frobenius structure, which is determined by a restricted set of data, obtained from the canonical solution of the Birkhoff problem for the Brieskorn lattice of f as defined in [4]. This is explained in [5] where examples are given. This paper generalizes this result to any f, up to the existence of a lattice in A_f (of course, made with subdiagram Laurent polynomials).

Our recipe to construct Frobenius manifolds is the following:

(a) find M. Saito's canonical solution of the Birkhoff problem for the Brieskorn lattice of f (it exists, thanks to [4]),
(b) starting from a solution as in (a), solve, and this step is algebraic, the Birkhoff problem for the Brieskorn lattice of an injective and surjective deformation of f (choose the one which makes the computations as simple as possible, see below): this will give the canonical Frobenius type structure (the initial condition),
(c) finally, use [8] to get a universal deformation of the Frobenius type structure given in (b). We then get the desired canonical Frobenius structure with the help of the primitive and homogeneous form ω.

In practice, the main difficulty is part (a): even if one computes a solution of the Birkhoff problem for the Brieskorn lattice of f (in the nondegenerate and convenient case it can be done using Kouchnirenko's division theorem and the relation between the V-filtration and the Newton filtration, see [4, Sect. 4]), it can be hard to decide if it is the canonical one given by M. Saito's method or not, although it can be done in some concrete situations (see [5], especially Proposition 5.2). In principle, (b) follows from (a) as shown in the example below. Last notice that the process in part (c) is not algebraic and can be difficult to carry out.

How to compute an initial condition $\mathbb{F}_o$, starting with a solution of the Birkhoff problem for the Brieskorn lattice of f? Here is an example. We will see that the multiplication by f on A_f *is not regular: this example does not enter in the framework of* [6]. All the following computations can be done as in [5], using [4, Sect. 4]. Let $f : \mathbb{C}^* \to \mathbb{C}$ be defined by

$$f(u) = u^{-2} + u^2.$$

One has $\mu(f) = 4$ and the spectrum of (G_0^o, G^o) is equal to $(0, \frac{1}{2}, \frac{1}{2}, 1)$. First, we compute a solution of the Birkhoff problem for G_0^o. Let $\varepsilon_1^o = [\frac{du}{u}]$, $\varepsilon_2^o = [\frac{du}{u^2}]$, $\varepsilon_3^o = [du]$, $\varepsilon_4^o = [udu]$ where [ ] denotes the class in G_0^o. Using the fact that $df = (-\frac{2}{u^3} + 2u)du$ and the formulas in Sect. 4.2, we find that the matrix of $\theta^2 \nabla_{\partial_\theta}$ in the basis

$$\varepsilon^o = (\varepsilon_1^o, \varepsilon_2^o, \varepsilon_3^o, \varepsilon_4^o)$$

of G_0^o over $\mathbb{C}[\theta]$ takes the form

$$A_0^o + A_\infty \theta,$$

where

$$A_0^o = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \quad A_\infty = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that A_0^o is not regular as its minimal polynomial is equal to $(X-2)(X+2)$. In the same way as in [5, Lemma 4.1], we put

$$S^o(\varepsilon_1^o, \varepsilon_4^o) = S^o(\varepsilon_2^o, \varepsilon_2^o) = S^o(\varepsilon_3^o, \varepsilon_3^o) \in \mathbb{C}^*$$

and $S^o(\varepsilon_i^o, \varepsilon_j^o) = 0$ otherwise. This defines the duality S^o and we check that $(A_0^o)^* = A_0^o$ and $A_\infty^* + A_\infty = I$, where * denotes the adjoint with respect to S^o. Notice now that (u) is a lattice in A_f: we choose the subdiagram deformation

$$F(u, x) = u^{-2} + u^2 + xu$$

and we compute, starting from ε^o, a solution of the Birkhoff problem for G_0, the Brieskorn lattice attached to F. The matrix of the connection ∇ in the basis $\varepsilon' = (\varepsilon_1', \varepsilon_2', \varepsilon_3', \varepsilon_4')$, where

$$\varepsilon_1' = \left[\frac{du}{u}\right], \varepsilon_2' = \left[\frac{du}{u^2}\right], \varepsilon_3' = [du], \varepsilon_4' = [udu]$$

([ ] denotes now the class in G_0), takes the form

$$\left(\frac{B_0(x)}{\theta} + B_\infty(x)\right)\frac{d\theta}{\theta} + \left(\frac{C_1(x)}{\theta} + C_2(x)\right)dx,$$

where

$$B_0(x) = \begin{pmatrix} 0 & \frac{3x}{2} & 0 & 2 \\ 0 & 0 & 2 & \frac{x}{2} \\ \frac{3x}{2} & 2 & 0 & 0 \\ 2 & 0 & \frac{x}{2} & -\frac{x^2}{4} \end{pmatrix}, \quad B_\infty(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{x}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$C_1(x) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \frac{x}{2} \end{pmatrix}, \quad C_2(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We get a flat basis (i.e. a solution of the Birkhoff problem for G_0 as in the proof of Theorem 4.3.1) putting

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (\varepsilon_1', \varepsilon_2', \varepsilon_3', \varepsilon_4')P(x)$$

where

$$P(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{x}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the basis $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, the matrix of the connection ∇ takes the form

$$\left(\frac{A_0(x)}{\theta} + A_\infty\right)\frac{d\theta}{\theta} + \frac{C(x)dx}{\theta},$$

where $A_\infty = \mathrm{diag}(0, 1/2, 1/2, 1)$. A duality S (a flat extension of S^o) is defined on G_0 by

$$S(\varepsilon_1, \varepsilon_4) = S(\varepsilon_2, \varepsilon_2) = S(\varepsilon_3, \varepsilon_3) = S^o(\varepsilon_1^o, \varepsilon_4^o) \in \mathbb{C}^*$$

and $S(\varepsilon_i, \varepsilon_j) = 0$ otherwise. We check that $A_0(x)^* = A_0(x)$ and $A_\infty^* + A_\infty = I$, where * denotes the adjoint with respect to S. Thanks to the proof of Theorem 4.3.1, we get an initial condition $\mathbb{F}_o$.

Notice that (u^{-1}) is also a lattice in A_f: if we start from the initial data given by the deformation

$$G(u, x) = u^{-2} + xu^{-1} + u^2$$

we obtain, thanks to Theorem 7.1.3, a Frobenius structure isomorphic to the one attached to F. In other words, one has the choice to define the initial data, of course, the idea is to start with a deformation F which makes the computations as simple as possible. Last, it follows from Theorem 2 that the canonical Frobenius structure attached by Theorem 1 to the Laurent polynomial

$$u^{-2} + 12u + u^2$$

can be deduced from the one attached to f.

Let us finish by noticing that the results of this paper remain true, up to slightly modifications and up to the existence of a canonical *homogeneous* pre-primitive form (this is much more restrictive), if one considers convenient and nondegenerate polynomial functions defined on $U = \mathbb{C}^n$: one has to take care that $\mathcal{N}_1 K$ does not always embed in

A_f. This is emphasized in [2, Sect. 4.2] and especially relevant in the quasi-homogeneous case where the multiplication by f on A_f is zero (and far from being semisimple and regular). An example is given in [2, Sect. 4.2.2]. The previous restrictions show the difficulties that one has to overcome if one wants, and it is inevitable, to extend the results of this paper to general regular tame functions, not necessarily convenient and nondegenerate.

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